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LATTICE STRUCTURE AND COMPUTATIONAL MODE: ADVECTION EQUATIONS

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The term "computational mode: was introduced by Platzman (1954) to identify a physically irrelevant eigen-frequency of the centered difference approximation of the advection equation

$$\frac{\partial \zeta}{\partial t} = - U \frac{\partial \zeta}{\partial x} \tag{1}$$

Subsequently, Matsuno (1966) demonstrated that the semi-discrete approximation of (1) in which the space derivative alone is replaced by a centered finite-difference approximation, permitted the appearance of a "spatial computational mode." This was a physically irrelevant, additional eigen-wave number. The term "computational mode" is therefore ambiguous.

In an attempt to clarify this notion, we shall follow-up a point made by Platzman (1958) regarding the structure of the centered difference approximation to equation (1). Using the symbol ζ_j^n, to identify the value of ζ at a grid-point for which $t = n\Delta t$ and $x = j\Delta x$, one may write for (1), the second order approximation:

$$\frac{\zeta_j^{n+1} - \zeta_j^{n-1}}{\Delta t} = - U \frac{\zeta_{j+1}^n - \zeta_{j-1}^n}{\Delta x} \tag{2}$$

In order to make clear the "lattice structure" of equation (2) it is convenient to develop from it the following form,

$$\frac{\zeta_j^{n+2} + \zeta_j^{n-2} - 2\,\zeta_j^n}{(\Delta t)^2} = U^2 \frac{\zeta_{j+2}^n + \zeta_{j-2}^n - 2\,\zeta_j^n}{(\Delta x)^2} \tag{3}$$

By inspection of the indices appearing in equation (3), one notes that they are all in one of the four classes:

Class 1: n, odd; j, odd
Class 2: n, even; j, odd
Class 3: n, odd; j, even
Class 4: n even; j, even (4)

One may visualize the grid-point mesh to be composed of four sub-meshes or lattices. On each one of these lattices, equation (3) applies quite independently of the other three lattices.

As pointed out by Platzman the equation (3) is an approximation to the wave equation and admits solutions in the form of wave functions,

$$\zeta_j^n = A\ e^{i\,(rj\Delta x\ -\ qn\Delta t)} \tag{5}$$

It is very significant to note that in (5) the wave number, r, and frequency, q, have cut-offs at the wavelength corresponding to $4\Delta x$ and the period corresponding to $4\Delta t$. More precisely r and q are confined to the intervals,

$$0 < \left| r\Delta x \right| < \frac{\pi}{2}$$

$$0 < \left| q\Delta t \right| < \frac{\pi}{2} \tag{6}$$

This follows from the structure of the grid lattice corresponding to each of the four classes in (4). Consequently, the frequency equation obtained when (5) is substituted into (3)

$$\sin q\Delta t = \pm \left(\frac{U\Delta t}{\Delta x}\right) \sin r\Delta x \tag{7}$$

possess only single valued functions of r and q over the allowable range (6).

This behavior may be contrasted with that resulting from the substitution of the wave function (5) into the equation (2). Since the solution to (2) is defined for all grid points, not just a particular lattice, the restriction (6) on r and q is not appropriate. Rather one may allow them to lie in the ranges,

$$0 < \left| r\Delta x \right| < \pi$$

$$0 < \left| q\Delta t \right| < \pi \tag{8}$$

This implies that the smallest wavelength resolvable is $2\Delta x$, and the smallest period is $2\Delta t$. The frequency equation relating r and q is

$$\sin q\Delta t = + U\frac{\Delta t}{\Delta x} \sin r\Delta x \tag{9}$$

One immediately notices that the sine function has a double valued structure over the intervals prescribed in (8). It is this multiple-value property which gives rise to the "computational mode" form used by Platzman and Matsuno.

The Computational Mode Representation of the Velocity

If one considers the system of linear equations governing a one-dimensional linear gravitational oscillation,

$$\frac{\partial u}{\partial t} = - g \frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial t} = - H \frac{\partial u}{\partial x} \tag{10}$$

and uses the centered finite difference approximation,

$$u_j^{n+1} - u_j^{n-1} = - g \frac{\Delta t}{\Delta x} (h_{j+1}^n - h_{j-1}^n)$$

$$h_j^{n+1} - h_j^{n-1} = - H \frac{\Delta t}{\Delta x} (u_{j+1}^n - u_{j-1}^n) \tag{11}$$

one may determine two necessary conditions for the existence of solutions of the form:

$$u_j^n = A\, e^{i\,(qn\Delta t + rj\Delta x)}$$

$$h_j^n = B\, e^{i\,(qn\Delta t + rj\Delta x)} \tag{12}$$

The first of these conditions is the linear stability criterion

$$g\, H(\Delta t)^2 < (\Delta x)^2 \tag{13}$$

The second, is the frequency equation

$$\sin q\Delta t = \pm (gH)^{1/2} \Delta t\, (\sin r\Delta x)/\Delta x \tag{14}$$

which must be satisfied by allowable values of q and r. Since the solutions (12) are to apply to the entire grid, the frequency and wave number must lie in the intervals (8). However, if we choose $\hat{q}$ and $\hat{r}$ both positive in the interval (6), the modes allowed by (14) may be tabulated as in Table 1. We have assumed that $\hat{q}$ and $\hat{r}$ satisfy

$$\sin \hat{q}\, \Delta t = + \frac{(gH)^{1/2} \Delta t}{\Delta x} \sin \hat{r}\, \Delta x \tag{15}$$

q	r	q	r
$-\hat{q}$	$\hat{r}$	$\frac{\pi}{\Delta t} + \hat{q}$	$\hat{r}$
$\hat{q}$	$-\hat{r}$	$\frac{\pi}{\Delta t} - \hat{q}$	$-\hat{r}$
$\hat{q}$	$\hat{r}$	$\frac{\pi}{\Delta t} - \hat{q}$	$\hat{r}$
$-\hat{q}$	$-\hat{r}$	$\frac{\pi}{\Delta t} + \hat{q}$	$-\hat{r}$
$-\hat{q}$	$\frac{\pi}{\Delta x} - \hat{r}$	$\frac{\pi}{\Delta t} + \hat{q}$	$\frac{\pi}{\Delta x} - \hat{r}$
$\hat{q}$	$\frac{\pi}{\Delta x} + \hat{r}$	$\frac{\pi}{\Delta t} - \hat{q}$	$\frac{\pi}{\Delta x} + \hat{r}$
$\hat{q}$	$\frac{\pi}{\Delta x} - \hat{r}$	$\frac{\pi}{\Delta x} - \hat{q}$	$\frac{\pi}{\Delta x} - \hat{r}$
$-\hat{q}$	$\frac{\pi}{\Delta x} + \hat{r}$	$\frac{\pi}{\Delta t} + \hat{q}$	$\frac{\pi}{\Delta x} + \hat{r}$

TABLE 1 -- The values of frequency q and wave number r which satisfy equation (14) expressed in terms of $\hat{r}$ and $\hat{q}$ defined in (15).

Using these allowable free modes the velocity, u_j^n, may be expressed,

$$u_j^n = \left[V + U_1 e^{i\pi n} + U_2 e^{i\pi j} + U_3 e^{i\pi(j+n)} \right] e^{i(\hat{r}j\Delta x \pm \hat{q}n\Delta t)} \tag{16}$$

The coefficient of the exponential wave function may be regarded as a scheme for superimposing the four solutions existing on each lattice defined in (4). Suppose, for example, that on lattice 1

$$u_j^n = A\, e^{i(\hat{r}j\Delta x \pm \hat{q}n\Delta t)} \tag{17}$$

then from (16) and (4) one has

$$A = \left[V - U_1 - U_2 + U_3 \right] \tag{18}$$

Finally, note that the wave number and frequency in (16) are relatively low. Therefore the exponential wave function is slowly variable in space-time.

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